LINEAR STOCHASTIC SYSTEMS, A WHITE NOISE APPROACH

Daniel Alpay *
Department of mathematics
Ben–Gurion University of the Negev
Beer-Sheva
Israel

Joint work with David Levanony
(Electrical Engineering dpt, Ben-Gurion University)

*Earl Katz Family chair in algebraic system theory

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See also

A-Levanony: *Rational functions associated to the white noise space and related topics.* **Potential Analysis, vol. 29 (2008) pp. 195-220.**

A-Levanony. *On the reproducing kernel Hilbert spaces associated with the fractional and bi–fractional Brownian motions.* **Potential Analysis, vol. 28 (2008), pp. 163-184.**

The classical case:

Discrete-time, time-invariant linear system

$$y_n = (h * u)_n = \sum_{m=0}^{n} h_{n-m} u_m, \qquad n = 0, 1, \dots$$

h_n **are pre-assigned complex numbers (***impulse response***)**

u_n **input sequence**

y_n **output sequence**

want to relate properties of (h_n) *or of the transfer function*

$$\boxed{\widehat{h}(\zeta) = \sum_{n=0}^{\infty} \zeta^n h_n}$$

to the input-output behavior

BIBO stability:

Theorem: *There is a $M > 0$ such that for all $(u_m) \in \ell_\infty(\mathbb{Z})$*

$$\sup_{n \in \mathbb{Z}} |y_n| \leq M \sup_{n \in \mathbb{Z}} |u_n|$$

if and only if

$$\sum_{n=0}^{\infty} |h_n| \leq M.$$

The system is called *dissipative* if the ℓ_2 norm of the output is always less or equal to the ℓ_2 norm of the input.

Theorem: *A linear system is time-invariant, causal and dissipative if and only if its transfer function is analytic and contractive in the open unit disk.*

In other words, the system has a transfer function which is a Schur function, or, equivalently, if the kernel

$$\boxed{\frac{1-\widehat{h}(\zeta)\widehat{h}(\nu)^*}{1-\zeta\nu^*}}$$

is positive in the open unit disk.

Aim of the work: *to allow randomness both in the impulse response and in the input, and obtain similar results (also for the continuous case)*

Problem: A Gaussian input into a linear system with random coefficients cannot be expected to result in a Gaussian output

We use the white noise space setting and replace the pointwise product by the Wick product, enabling Gaussian input-output relations when the underlying system has random coefficients.

This has the advantage of preserving the Gaussian input-output relation, while allowing uncertainty in the form of randomness in the linear system under study.

Strategy: We now consider

$$y_n = \sum_{m \in \mathbb{Z}} h_{n-m} \lozenge u_m, \quad n \in \mathbb{Z},$$

where the y_n, u_n and h_n are now random variables in the *white noise space*, or more generally, in the *Kondratiev space* (or more precisely, in some Hilbert subspace of it), and where $\lozenge$ denote the *Wick product*.

When one of the factors is not random, the Wick product reduces to the pointwise product.

The *Hermite transform* allows to translate these problems into problems for analytic functions (analytic in a countable number of variables).

The white noise space and the Kondratiev space

The function

$$\boxed{K(s_1 - s_2) = \exp(-\|s_1 - s_2\|^2_{\mathbf{L}_2(\mathbb{R})}/2)}$$

is positive on the Schwartz space $\mathcal{S}$ of rapidly decreasing, infinitely differentiable functions.

$\mathcal{S}$ is a nuclear space, and the Bochner-Minlos theorem ensures the existence of a probability measure P on the Borel sets $\mathcal{F}$ of $\Omega = \mathcal{S}'$ such that

$$\int_\Omega e^{i\langle s,\omega\rangle_{\mathcal{S},\mathcal{S}'}} dP(\omega) = \exp(-\|s\|^2_{\mathbf{L}_2(\mathbb{R})}/2).$$

$$\boxed{\textit{The white noise space is } \mathcal{W} = \mathbf{L}_2(\Omega, \mathcal{F}, P).}$$

The white noise space is a convenient setting to define Brownian motion. An important feature here is the Wick product.

ℓ: denotes the set of sequences of integers $(n_0, n_1, \ldots)$ which are zero after a certain stage.

An orthogonal basis of the white noise space is given by the Hermite functions $(H_\alpha)_{\alpha\in\ell}$. These functions are computed in terms of the Hermite polynomials,

Every element in $\mathcal{W}$ can be written as

$$F(\omega) = \sum_{\alpha \in \ell} c_\alpha H_\alpha(\omega), \qquad c_\alpha \in \mathbb{C},$$

with

$$\|F\|^2_{\mathcal{W}} = \sum_{\alpha \in \ell} |c_\alpha|^2 \alpha! < \infty$$

The Wick product is defined through the Hermite functions by

$$H_\alpha \lozenge H_\beta = H_{\alpha+\beta}, \qquad \alpha, \beta \in \ell.$$

It is independent of the chosen basis in the white noise space

In general, the Wick product of two elements in the white noise space need not be in the white noise space. The most convenient space which is stable with respect to the Wick product is the Kondratiev space S_{-1}.

To define S_{-1} we first introduce for $k \in \mathbb{N} = \{1, 2, \ldots\}$ the Hilbert space $\mathcal{H}_k$ which consist of series in the H_α such that

$$\|f\|_k \stackrel{\text{def.}}{=} \left(\sum_{\alpha \in \ell} |c_\alpha|^2 (2\mathbb{N})^{-k\alpha} \right)^{1/2} < \infty.$$

The Kondratiev space S_{-1} is the inductive limit of the spaces $\mathcal{H}_k$.

When either one of the factors f or g in S_{-1} is nonrandom, the Wick product $f \lozenge g$ reduces to the pointwise product fg.

Important point: There is an isomorphism between the white noise space and the reproducing kernel Hilbert space with reproducing kernel

$$\boxed{\exp \langle z, w\rangle_{\ell_2}}$$

Let $z = (z_1, z_2, \ldots) \in \mathbb{C}^{\mathbb{N}}$**. The linear map**

$$\mathrm{I}(H_\alpha) = z^\alpha$$

extends to a unitary map between the white noise space and the Fock space. The map I **is called the** *Hermite transform*.

Note that the the Hermite transform is a unitary mapping from $\mathcal{H}_k$ **onto the reproducing kernel Hilbert space with reproducing kernel**

$$K_k(z, w) = \sum_{\alpha \in \ell} z^\alpha (w^*)^\alpha (2\mathbb{N})^{k\alpha},$$

Applying the Hermite transform to

$$y_n = \sum_{m=0}^{n} h_{n-m} \Diamond u_m, \qquad n \in \mathbb{N},$$

leads to the following: let

$$y_n(\omega) = \sum_{\alpha \in \ell} y_\alpha(n) H_\alpha(\omega)$$

and

$$h_n(\omega) = \sum_{\alpha \in \ell} h_\alpha(n) H_\alpha(\omega),$$

where the coefficients $y_\alpha(n)$ and $h_\alpha(n)$ are nonrandom complex numbers. Then,

$$y_n = \sum_{\alpha \in \ell} H_\alpha(\omega) \left(\sum_{m \in \mathbb{Z}} \sum_{\beta \leq \alpha} h_{\alpha-\beta}(n-m) u_\beta(m) \right),$$

After taking the Hermite transform

$$\mathrm{I}(y_n) = \sum_{\alpha \in \ell} z^{\alpha} \left(\sum_{m=0}^{n} \sum_{\beta \leq \alpha} h_{\alpha-\beta}(n-m) u_{\beta}(m) \right),$$

$$\boxed{y_{\alpha}(n) = \sum_{m=0}^{n} \sum_{\beta \leq \alpha} h_{\alpha-\beta}(n-m) u_{\beta}(m)}$$

We have two convolutions: the first is with respect to the index in ℓ, which is related to the stochastic aspect of the system; the second is with respect to the time variable.

The $\mathscr{Z}$ transform (denoted by $\widehat{y}$, with variable ζ) then leads to

$$\begin{aligned}\widehat{y}(\zeta, z) &\overset{\text{def.}}{=} \sum_{n\in\mathbb{Z}} \mathbf{I}(y_n)\zeta^n \\ &= \sum_{\alpha\in\ell} z^{\alpha} \sum_{\beta\leq\alpha} \widehat{h}_{\alpha-\beta}(\zeta)\widehat{u}_{\beta}(\zeta) \\ &= \left(\sum_{\alpha\in\ell} z^{\alpha}\widehat{h}_{\alpha}(\zeta)\right)\left(\sum_{\alpha\in\ell} z^{\alpha}\widehat{u}_{\alpha}(\zeta)\right).\end{aligned}$$

Definition: *The function*

$$\boxed{\mathscr{H}(\zeta, z) = \sum_{\alpha\in\ell} z^{\alpha}\widehat{h}_{\alpha}(\zeta) = \sum_{n\in\mathbb{Z}} \zeta^n(\mathbf{I}(h_n))(z)}$$

is the called the generalized transfer function of the system.

When all $\widehat{h}_{\alpha}(\zeta) = 0$ for $\alpha \neq 0$, we retrieve the classical notion of the transfer function. We can thus define a hierarchy of systems, depending on the properties of the function $\mathscr{H}(\zeta, z)$.

BIBO stable linear discrete time stochastic systems.

Fix some integer $l > 0$**, and let** $k > l + 1$**. Consider** $h \in \mathcal{H}_l$ **and** $u \in \mathcal{H}_k$**.**

Våge's inequality

$$\|h \Diamond u\|_k \leq A(k-l)\|h\|_l \|u\|_k,$$

where

$$A(k-l) = \sum_{\alpha \in \ell} (2\mathbb{N})^{(l-k)\alpha}.$$

The above inequality expresses the fact that the multiplication operator

$$T_h : u \mapsto h \Diamond u$$

is a bounded map from the Hilbert space $\mathcal{H}_k$ **into itself, and that its operator norm** $\|T_h\|_{\mathrm{op},l,k}$ **satisfies the inequality**

$$\|T_h\|_{\mathrm{op},l,k} \leq A(k-l)\|h\|_l.$$

We set

$$\|T_h\|_{\mathrm{op},l,k} = \|T_h\|.$$

Definition: *A random discrete time signal will be a sequence* (u_n) *of elements in the Kondratiev space, such that there exists a* $k \in \mathbb{N}$ *(depending on the signal) such that*

$$u_n \in \mathcal{H}_k, \quad \forall n \in \mathbb{N}.$$

Note that k is imposed to be independent of n.

Theorem: Let $k > l + 1$ and let (h_n) be a sequence of elements in $\mathcal{H}_l$ indexed by $\mathbb{N}$. Then
(a) **The sums**

$$y_n = \sum_{m=0}^{n} h_{n-m} \Diamond u_m, \qquad n \in \mathbb{N},$$

converge absolutely in $\mathcal{H}_k$ for all inputs $(u_m)_{m\in\mathbb{N}}$ such that $\sup_{m\in\mathbb{N}} \|u_m\|_k < \infty$, and
(b) **There exists an $M > 0$ such that, for all such inputs $(u_n)_{n\in\mathbb{N}}$, it holds that**

$$\sup_{n\in\mathbb{N}} \|y_n\|_k \leq M \sup_{n\in\mathbb{N}} \|u_n\|_k$$

if and only if for all $v \in \mathcal{H}_k$ with $\|v\|_k = 1$ it holds that

$$\sum_{n\in\mathbb{N}} \|T^*_{h_n}(v)\|_k \leq M.$$

In the nonrandom case, where the h_n are (nonrandom) complex numbers, the Wick product reduces to a pointwise product, and we have classical convolution systems. Furthermore, in this case,

$$\|T_{h_n}^{*} v\|_k = |h_n| \cdot \|v\|_k,$$

and we retrieve the well known BIBO stability condition

The condition

$$\sum_{n \in \mathbb{N}} \|T_{h_n}\| \leq M$$

on the norms of the operators T_{h_n} implies condition in the theorem.

ℓ_1-ℓ_2 **stable random systems**

Theorem: *Let $l > k + 1$ and assume that the $h_n \in \mathcal{H}_l$. Then there exists $M > 0$ such that*

$$\left(\sum_{n=0}^{\infty} \|y_n\|_k^2\right)^{1/2} \leq M \sum_{n=0}^{\infty} \|u_n\|_k$$

for all inputs (u_n) such that the right handside of the above equation is finite, if and only if its transfer function belongs to $\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_l)$, i.e. if and only if there exists a number $c > 0$ such that the kernel

$$\frac{K_l(z,w)}{1 - \zeta\nu^*} - c\mathscr{H}(\zeta, z)\mathscr{H}(\nu, w)^*$$

is positive in $\mathbb{D} \times \mathbb{K}_l$.

Dissipative discrete time random systems

Theorem: *Let $k \in \mathbb{N}$. The operators M_{z_j} are bounded from $\mathcal{H}(K_k)$ into itself. A linear operator from $\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_k)$ into itself is contractive and such that*

$$T(M_\zeta f) = M_\zeta T f$$
$$T(M_{z_j} f) = M_{z_j} T f$$

if and only if it is of the form

$$(Tf)(\zeta, z) = \mathscr{S}(\zeta, z) f(\zeta, z)$$

where $\mathscr{S}$ is such that the kernel

$$(1 - \mathscr{S}(\zeta, z)\mathscr{S}(\nu, w)^*) \frac{K_k(z, w)}{(1 - \zeta\nu^*)}$$

is positive in $\mathbb{D} \times \mathbb{K}_k$.